

Fractional Derivative Cosmology.

Mark D. Roberts,
54 Grantley Avenue, Womersh Park, GU5 0QN, UK
mdr@ihes.fr

6th of September 2009

Abstract

The degree by which a function can be differentiated need not be restricted to integer values. Usually most of the field equations of physics are taken to be second order, curiosity asks what happens if this is only approximately the case and the field equations are nearly second order. For Robertson-Walker cosmology there is a simple fractional modification of the Friedman and conservation equations. In general fractional gravitational equations similar to Einstein's are hard to define as this requires fractional derivative geometry. What fractional derivative geometry might entail is briefly looked at and it turns out that even asking very simple questions in two dimensions leads to ambiguous or intractable results. A two dimensional line element which depends on the Gamma-function is looked at.

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1 Introduction

1.1 Motivation

The field equations of fundamental physics are usually taken to be second order. Like any other property of a physical theory this should be subjected to experimental and observational tests to see what the experimental bounds are. A method of examining this is by using fractional derivatives to investigate the properties of differential equations which are almost second order. Electromagnetism might produce the best tests of how near to second order governing field equations need to be, but here an attempt is made to see what modification of gravity theory can be constructed using fractional derivatives.

1.2 History

In 1695 L'Hôpital wrote to Leibniz asking him what happened if the number of times a function was differentiated was not an integer but rather $\frac{1}{2}$, from this early beginning the subject of fractional derivatives was born. Although the derivatives are called fractional derivatives they can take any real or sometimes complex value. Recent textbooks include [3, 7]. Recent applications to physics include construction of a fraction Schrödinger equation [6, 4] and some properties of field theories [1, 11].

1.3 Methodology

There are two distinct methods of approaching what fractional derivative cosmology could be. The simplest is *last step modification* in which Einstein's field equations for a given geometric configuration are replaced with analogous fractional field equations, in other words $\partial_a \rightarrow \mathcal{D}_a^k$ after the field equations for a specific geometry have been derived. The fundamentalist methodology is *first step modification* in which one starts by constructing fractional derivative geometry. The problem with the last step approach is that it probably only gives consistent answers for geometric configurations that are expressed in rectilinear coordinates; the problem with the first step approach is that the whole of geometry has to be rethought through, even things like linear coordinate transformations have to be replaced by fractional ones, perhaps quadratic forms by fractional forms and so on; intermediate step approaches seem to have the disadvantages of both of the above approaches. When using fractional derivatives it is not always the case that differentiating a constant gives zero see Podlubny [7]p.80, so it

is not clear what kind of fractional derivative to use, here a pragmatic approach is adopted: if a fractional derivative can be made to work then it is used.

1.4 Outline & Conventions

In §2 an attempt is made to see how the newtonian $\frac{1}{r}$ potential is modified when the d’Almbertian is replaced at the ‘last step’ by fractional derivatives, it is found that the ‘last step’ assumption is not necessarily correct as spherically flat spacetime need not take its familiar form. In §3 Robertson-Walker cosmology is looked at in the last step approach. In §4 briefly looks at what fractional derivative geometry might entail. In §5 is the conclusion. p is used for degree of fractional differentiation, the familiar non-fractional differential equations are recovered when $p = 1$, \mathcal{P}, μ, U_a are the pressure, density and co-moving vector of a fluid. Calculations were carried out using maple9/grtensorII [5].

2 Newtonian Gravity

Vacuum Newtonian gravity is taken to be governed by the D’Almbertian operator acting on a scalar function

$$\square\phi = 0, \tag{1}$$

Applying the last step fractional derivative substitution approach for spherical symmetry gives

$$\mathcal{D}^{2p}\psi + \frac{2}{r}\mathcal{D}^p\psi = 0 \tag{2}$$

numerically solving this for values close to one produces potentials as illustrated in figure one. For p close to one the potential can be made arbitrarily close to the non-fractional case. Rates of decay of potentials have been discussed at length in [8]. At first sight one might imagine that all that now needs doing is using either expansions or numerical analysis to compare this result with observations: but there is there is a serious problem with equation 2, the $\frac{2}{r}$ term assumes a standard spherically symmetric flat background spacetime, however for fractional derivatives this assumption does not necessarily hold, the two-sphere in spherical coordinates is not necessarily the familiar one as this requires assumptions of standard geometry which might not hold in fractional geometry, see §4.2.

3 Fractional Robertson-Walker Cosmology

3.1 Friedman Equation

For the moment we retain the standard quadratic form of the Robertson-Walker line element

$$ds^2 = -dt^2 + \frac{a(t)^2}{(1 + k(x^2 + y^2 + z^2)/4)^2} (dx^2 + dy^2 + dz^2), \quad (3)$$

where $k = 0, \pm 1$. The stress is taken to be that of a perfect fluid

$$T_{ab} = (\mu + \mathcal{P})U_a U_b + \mathcal{P}g_{ab}, \quad U_a = (1, 0, 0, 0), \quad (4)$$

which for present purposes contains no derivatives, this might change if the Clebsch representation of the comoving vector field U_a is used. For general relative the dynamics are governed by the Friedman equation and the first conservation equation $U_a T_{\dots;b}^{ab} = 0$ replacing partial derivatives with fractional derivatives at the last moment the Friedman and conservation equations become

$$3 \left[k + (\mathcal{D}_t^p a)^2 \right] = \kappa \mu a^2, \quad a^3 \mathcal{D}_t^p \mathcal{P} = \mathcal{D}_t^p [a^3 (\mu + \mathcal{P})], \quad (5)$$

respectively. For the simplest non-fraction $p = 1$ case take both $k, \mathcal{P} = 0$ then the conservation equation integrates to

$$\mu = \frac{C}{a^3}, \quad (6)$$

and then the Friedman equation integrates to give

$$a = \left(\frac{\kappa C}{3} t \right)^{\frac{2}{3}}. \quad (7)$$

For the $k \mp 1$ cases powers of t are replaced by trigonometric or hyperbolic functions.

3.2 Naive Approach

A naive generalization of the above non-fractional approach is just to replace the time derivative in the Friedman equation with a fractional time derivative. The Riemann-Liouville fractional derivative with lower terminal at 0, after correcting a typo [7] page 310 is

$$\mathcal{D}_t^p t^\nu = \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 1 - p)} t^{\nu - p}, \quad (8)$$

Absorbing constants into C gives the scale factor and density

$$a = Ct^{\frac{2p}{3}}, \quad \mu = C^{-2}t^{-2p}, \quad (9)$$

which is what might have been anticipated. There is a problem with the above in integrating the conservation equation 6 it was assumed that differentiating a constant gives zero but 8 gives

$$\mathcal{D}_t^p B = \frac{B}{\Gamma(1-p)} t^{-p}, \quad (10)$$

which is non-zero for non-zero B .

3.3 Compensating Pressure.

To rectify this a compensating pressure can be used, substituting 9 back in to 5 a pressure which satisfies the conservation equation is

$$\mathcal{P} = At^{-2p}, \quad A^{-1}C^{-2} = \frac{\Gamma(1-p)\Gamma(1-2p)}{\Gamma(1-3p)} - 1, \quad (11)$$

AC^2 is plotted in figure two. This shows that the compensating pressure is smaller than the density.

3.4 γ -equation of state.

The solution 11 is an example of a γ -equation of state for which

$$p = (\gamma - 1)\mu, \quad (12)$$

for 11 $\gamma = AC^2 + 1$. To investigate if 11 can be generalized choose 12 and

$$a = At^m, \quad \mu = Ct^n, \quad (13)$$

then the Friedman equation restricts the values of n and C

$$n = -2p, \quad C = \frac{3}{\kappa} \frac{\Gamma(m+1)^2}{\Gamma(m+1-p)^2}, \quad (14)$$

and the conservation equation places the restriction

$$\frac{(\gamma-1)\Gamma(1-2p)}{\gamma\Gamma(1-3p)} = \frac{\Gamma(3m-2p+1)}{\Gamma(3m-3p+1)}, \quad (15)$$

because m occurs inside a Γ -function it is hard to access the use of this solution. Generalizing 13 with $\mathcal{P} = Bt^r$ the conservation equation gives $n = r$ so that the results are the same.

4 Fractional Derivative Geometry

4.1 Fractional alterations

A generalized Christoffel symbol can be taken to be

$$\Gamma_{bc}^a = \frac{1}{2}g^{ad} \left(\mathcal{D}_c g_{bd} + \mathcal{D}_b g_{cd} - \mathcal{D}^d g_{bc} \right), \quad (16)$$

where \mathcal{D}_c is an operator with one spacetime index, in particular it can be a vector field, or the familiar partial differential operator, or a fractional derivative. A generalized Riemann tensor can be taken to be

$$GR_{bcd}^a = \mathcal{D}_c \Gamma_{db}^a - \mathcal{D}_d \Gamma_{cb}^a + \Gamma_{cf}^a \Gamma_{db}^f - \Gamma_{df}^a \Gamma_{cb}^f, \quad (17)$$

where for simplicity the operator \mathcal{D}_c is chosen to be the same one as in the generalized christoffel symbol. The fractional quadratic form could be

$$ds^{2p} = g_{ij} (dx^i)^p (dx^j)^p, \quad (18)$$

for $p = Z_+/2$ the metric requires $2 \times \alpha$ indices, otherwise the number of indices is not integer, the meaning of a metric with non-integer number of indices is not clear. Forms of the form 18 can occur in Finsler geometry [2], however in Finsler geometry there are still linear coordinate transformations rather than fractional ones of the form 24.

4.2 Fractional Flat Space

It is now possible to ask what is fractional rectilinear flat spacetime, for simplicity working in two dimensions with the usual quadratic form

$$ds^2 = -N(t, x)^2 dt^2 + F(t, x)^2 dx^2, \quad (19)$$

if the generalized Christoffel symbol 16 vanishes then the generalized Riemann tensor 17 will also vanish, looking at one component of the generalized Christoffel symbol

$$\Gamma_{tt}^t = \frac{1}{2N^2} \mathcal{D}_t N^2 \quad (20)$$

this will not vanish unless either $N = 0$ in which case 19 degenerates or caputso derivatives are used which results in the familar two dimensional rectilinear flat space with $N = F = 1$. The next question is what is fractional spherical flat spacetime. In two dimensions this question is what is the analog of the following: start with line element

$$ds^2 = dx^2 + dy^2, \quad (21)$$

and then perform the transformation

$$x = r \sin(\theta), \quad y = r \cos(\theta), \quad (22)$$

to give the line element

$$ds^2 = dr^2 + r^2 d\theta^2. \quad (23)$$

For the fractional case start with 21 and use coordinate transformations

$$dx^i = \mathcal{D}_i^p x^i dx^i, \quad (24)$$

choosing 22 to be replaced by

$$x = r^p \sin(\theta), \quad y = r^p \cos(\theta), \quad (25)$$

gives the line element

$$ds^2 = \Gamma(p+1)^2 dr^2 + 2\Gamma(p+1)r^p \cos\left(\frac{\pi p}{2}\right) dr d\theta + r^{2p} d\theta^2. \quad (26)$$

Altering the power of the trigonometric parts of 25 does not seem to remove the cross term but rather makes things more complicated. Maple/grtensorII [5] works out the standard, non-fractional derivative curvature to be

$$\begin{aligned} R_{r\theta r\theta} &= -p(p-1)r^{2(p-1)}, \quad R = \frac{2(p-1)}{p\Gamma(p)^2 \sin(\pi p/2)^2}, \\ R_{ab} &= \frac{1}{2}Rg_{ab}, \quad G_{ab} = 0, \quad RiemSq = R^2, \quad RicciSq = \frac{1}{2}R^2. \end{aligned} \quad (27)$$

Like all two dimensional line elements this solution obeys the vacuum Einstein equations. For fixed constant p it has constant curvature. Using the method of signature constants [9] the signature of 26 gives the same curvature in any combination. Two dimensional line elements have been studied by Witten [10]. Whether the line element 26 is explicitly flat using the curvature 17 leads to a calculation which has so far proved intractable.

4.3 Fractional geodesics.

Instead of starting with 16 and 17 one could attempt a more systematic approach by starting with a point particle lagrangian

$$\mathcal{L} = \sqrt{-\dot{x}^2} \rightarrow \left[- \left(\frac{dx^i}{d\tau} \right)^p \left(\frac{dx_i}{d\tau} \right)^p \right]^{\frac{1}{2p}}, \quad (28)$$

and then varying the fractional alteration to form a geodesic equation. 28 can also be thought of as the integral form of an element of arc, compare equation one [2]. Once one has a geodesic equation one has a connection. From the connection the commutation of covariant derivatives will give curvature. The problem with this is that it is not clear what the right hand side of 28 means, in principle it is not necessary to know explicitly what 28 means only to know what the variation of 28 is, this is unclear because terms such as $(\frac{dx^i}{d\tau})^p$ occur and there appears to be no way of defining them. The same problems seem to occur starting from two other forms of the point particle lagrangian such as the second-order form and the Hamiltonian form.

5 Conclusion

It is possible to produce fractional generalizations of Newtonian mechanics and Friedman-Robertson-Walker cosmology by replacing partial derivatives by fractional derivatives in familiar equations. In cosmology the degree of fraction differentiation p itself could be made a time dependent variable, although this was not looked at here. The fractional derivative results are what might have been anticipated: however for consistency one should start with fractional derivatives at the first step and this leads to the subject of fractional derivative geometry. What fractional derivative geometry should look like is unclear, here a first attempt was made at guessing what curvature and line elements might look like in two dimensions. By chance this lead to a line element depending on the Gamma-function.

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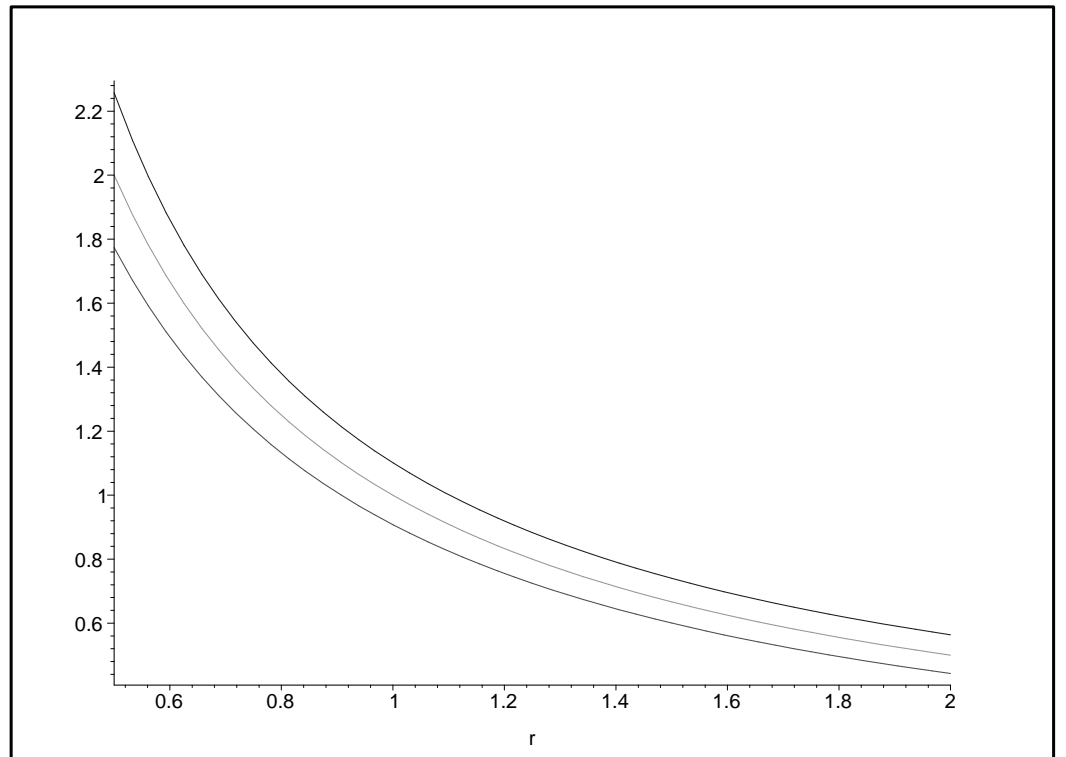


Figure 1: lowest line= $2 \cdot 10^4 \exp(-r^{0.1}/0.1)$, middle line= $1/r$, top line= $5 \cdot 10^{-5} \exp(r^{-0.1}/0.1)$

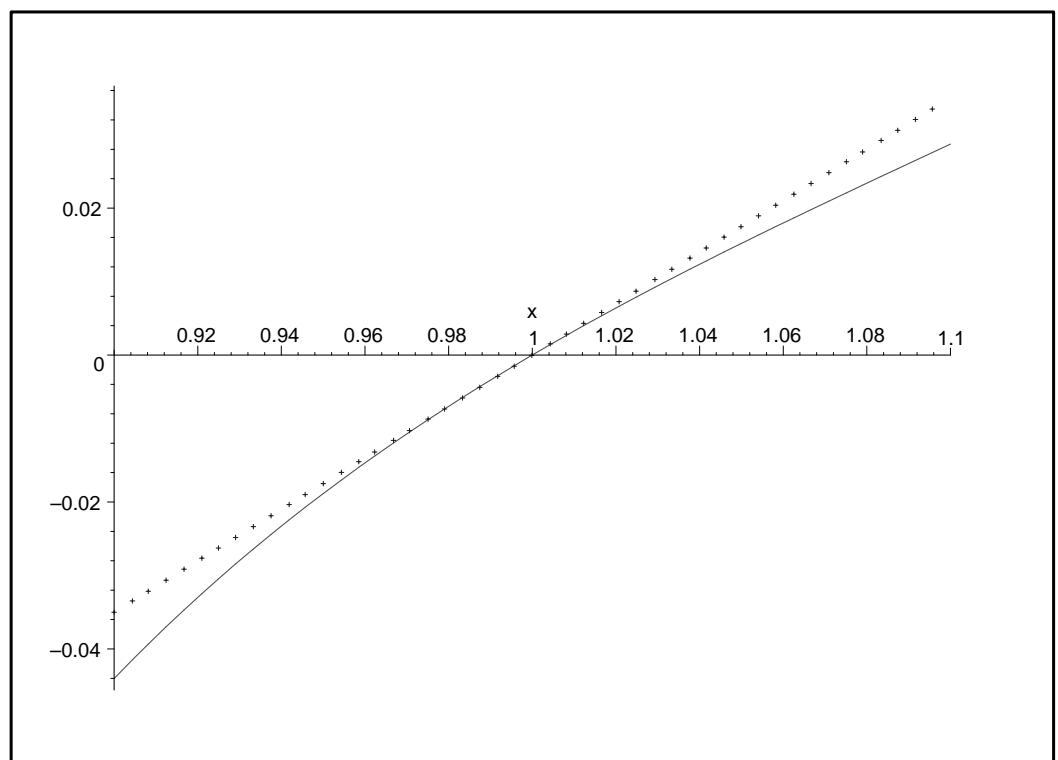


Figure 2: Red line is AC^2 , blue points are $y = 0.35(x - 1)$